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Some Elliptic Function Formulae.

By Thomas Craig, *Johns Hopkins University.*

On page 102 of Prof. Cayley's Treatise on Elliptic Functions occur the formulae for the differentiation of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ with respect to the modulus k; these are

$$\frac{d \operatorname{sn} u}{dk} = -\frac{k}{k'^2} \operatorname{cn} u \operatorname{dn} u \int_0 \operatorname{cn}^2 u \, du + \frac{k}{k'^2} \operatorname{sn} u \operatorname{cn}^2 u,$$

$$\frac{d \operatorname{cn} u}{dk} = \frac{k}{k'^2} \operatorname{sn} u \operatorname{dn} u \int_0 \operatorname{cn}^2 u \, du - \frac{k}{k'^2} \operatorname{sn}^2 u \operatorname{cn} u,$$

$$\frac{d \operatorname{dn} u}{dk} = \frac{k^3}{k'^2} \operatorname{sn} u \operatorname{cn} u \int_0 \operatorname{cn}^2 u \, du - \frac{k}{k'^2} \operatorname{sn}^2 u \operatorname{dn} u.$$

The following three forms for these differential coefficients may be obtained from these equations, or they may be obtained directly in a slightly different manner. The formulae may have been given before, but I do not remember to have seen them; they are, however, sufficiently interesting to make a note of them here. Starting with the relation

$$\frac{d}{du} \log \operatorname{dn} u = -k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u},$$

drop the factor $-k^2$ and differentiate this a second time; we have thus on reduction,

$$\frac{d}{du} \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = -\operatorname{sn}^2 u + \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}.$$

Subtract this from $\operatorname{cn}^2 u$ and we have

$$\operatorname{cn}^2 u - \frac{d}{du} \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = 1 - \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u},$$

or

$$\operatorname{cn}^2 u - \frac{d}{du} \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} = \frac{k'^2 \operatorname{sn}^2 u}{\operatorname{dn}^2 u};$$

and on integration

A $$k'^2\int_0 \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u} = \int_0 \operatorname{cn}^2 u\, du - \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

But

$$\int_0 \operatorname{cn}^2 u\, du = u - \int_0 \operatorname{sn}^2 u\, du =$$

$$u - \frac{1}{k^2}\left\{\frac{\Theta'(u)}{\Theta(u)} - u\left(1 - \frac{E}{K}\right)\right\} = \frac{1}{k^2}\left\{\left(\frac{E}{K} - k'^2\right)u + \frac{\Theta'(u)}{\Theta(u)}\right\}.$$

Assume now $\phi = \operatorname{am} u$ and write

$$u = \int_0 \frac{d\varphi}{\sqrt{1 - k^2 \operatorname{sn}^2 \varphi}};$$

then

$$\frac{du}{dk} = \int_0 \frac{k \operatorname{sn}^2 \varphi d\varphi}{(1 - k^2 \operatorname{sn}^2 \varphi)^{\frac{3}{2}}},$$

or

$$\frac{du}{dk} = \int_0 \frac{k \operatorname{sn}^2 u\, du}{\operatorname{dn}^2 u}.$$

Equation A now becomes

A′ $$kk'^2 \frac{du}{dk} = \left\{\left(\frac{E}{K} - k'^2\right)u + \frac{\Theta'(u)}{\Theta(u)}\right\} - k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

Now

$$\frac{d \operatorname{sn} u}{dk} = -\operatorname{cn} u \operatorname{dn} u \frac{du}{dk},$$

$$\frac{d \operatorname{cn} u}{dk} = \operatorname{sn} u \operatorname{dn} u \frac{du}{dk},$$

$$\frac{d \operatorname{dn} u}{dk} = k^2 \operatorname{sn} u \operatorname{cn} u \frac{du}{dk};$$

substituting from A′ for $\frac{du}{dk}$ we have the following formulae:

$$\frac{d \operatorname{sn} u}{dk} = \frac{-1}{kk'^2} \operatorname{cn} u \operatorname{dn} u \left\{\left(\frac{E}{K} - k'^2\right)u + \frac{\Theta'(u)}{\Theta(u)}\right\} + \frac{k}{k'^2} \operatorname{sn} u \operatorname{cn}^2 u,$$

B $$\frac{d \operatorname{cn} u}{dk} = \frac{1}{kk'^2} \operatorname{sn} u \operatorname{dn} u \left\{\left(\frac{E}{K} - k'^2\right)u + \frac{\Theta'(u)}{\Theta(u)}\right\} - \frac{k}{k'^2} \operatorname{sn}^2 u \operatorname{cn} u,$$

$$\frac{d \operatorname{dn} u}{dk} = \frac{k}{k'^2} \operatorname{sn} u \operatorname{cn} u \left\{\left(\frac{E}{K} - k'^2\right)u + \frac{\Theta'(u)}{\Theta(u)}\right\} + \frac{k^3}{k'^2} \frac{\operatorname{sn}^2 u \operatorname{cn}^2 u}{\operatorname{dn} u}.$$

By means of the formulae

$$\operatorname{sn} u = e^{-\frac{\pi}{4K}(K'-2iu)} \frac{-i}{\sqrt{k}} \Theta(u+iK') \div \Theta(u)$$

$$\operatorname{cn} u = e^{-\frac{\pi}{4K}(K'-2iu)} \sqrt{\frac{k'}{k}} \Theta(u+K+iK') \div \Theta(u)$$

$$\operatorname{dn} u = \sqrt{k'}\, \Theta(u+K) \div \Theta(u),$$

these may be written in the forms

$$\frac{d \operatorname{sn} u}{dk} = -e^{\frac{*}{4}} \frac{1}{k'k\sqrt{k}} \frac{\theta(u+K+iK')\,\theta(u+K)}{\theta^2(u)} \left\{ \left(\frac{E}{K} - k'^2 \right) u + \frac{\theta'(u)}{\theta(u)} \right\}$$
$$- e^{\frac{3*}{4}} \frac{i}{k\sqrt{kk'}} \frac{\theta(u+iK')\,\theta^2(u+k+iK')}{\theta^3(u)}$$

$$\text{B}' \qquad \frac{d \operatorname{cn} u}{dk} = -e^{\frac{*}{4}} \frac{i}{kk'\sqrt{kk'}} \frac{\theta(u+K)\,\theta(u+iK')}{\theta^2(u)} \left\{ \left(\frac{E}{K} - k'^2 \right) u + \frac{\theta'(u)}{\theta(u)} \right\}$$
$$- e^{\frac{3*}{4}} \frac{\sqrt{k}}{k'} \frac{\theta^2(u+iK')\,\theta(u+K+iK')}{\theta^3(u)}$$

$$\frac{d \operatorname{dn} u}{dk} = -e^{\frac{*}{2}} \frac{i}{k'\sqrt{k'}} \frac{\theta(u+iK')\,\theta(u+K+iK')}{\theta^2(u)} \left\{ \left(\frac{E}{K} - k'^2 \right) u + \frac{\theta'(u)}{\theta(u)} \right\}$$
$$- e^{*} \frac{k^2}{k'\sqrt{k'}} \frac{\theta^2(u+iK')\,\theta^2(u+K+iK')}{\theta(u+K)\,\theta^3(u)}$$

where for brevity * is written to denote $-\frac{\pi}{K}(K'-2iu)$.

Substituting in equation A the value of $\int \operatorname{cn}^2 u\, du$ we have

$$\int \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u} du = \frac{1}{k^2 k'^2} \left\{ \left(\frac{E}{K} - k'^2 \right) u + \frac{\theta'(u)}{\theta(u)} \right\} - \frac{1}{k'^2} \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

We can easily find eight other relations of this character, *i. e.* we can get the group of nine integrals,

$$\int \frac{1}{\operatorname{sn}^2 u} du, \quad \int \frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u} du, \quad \int \frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u} du,$$

$$\int \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u} du, \quad \int \frac{1}{\operatorname{cn}^2 u} du, \quad \int \frac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u} du;$$

$$\int \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u} du, \quad \int \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u} du, \quad \int \frac{1}{\operatorname{dn}^2 u}.$$

These together with

$$\int \mathrm{sn}^2 u\, du\,, \int \mathrm{cn}^2 u\, du\,, \int \mathrm{dn}^2 u\, du$$

make a series of twelve integrals of which the differential coefficients relatively to u are the squares of the twelve quantities

C

$$\begin{array}{ccc} \mathrm{sn}\, u\,, & \mathrm{cn}\, u\,, & \mathrm{dn}\, u \\ \dfrac{1}{\mathrm{sn}\, u}\,, & \dfrac{\mathrm{cn}\, u}{\mathrm{sn}\, u}\,, & \dfrac{\mathrm{dn}\, u}{\mathrm{sn}\, u} \\ \dfrac{\mathrm{sn}\, u}{\mathrm{cn}\, u}\,, & \dfrac{1}{\mathrm{cn}\, u}\,, & \dfrac{\mathrm{dn}\, u}{\mathrm{cn}\, u} \\ \dfrac{\mathrm{sn}\, u}{\mathrm{dn}\, u}\,, & \dfrac{\mathrm{cn}\, u}{\mathrm{dn}\, u}\,, & \dfrac{1}{\mathrm{dn}\, u} \end{array}$$

whose integrals with respect to u have been given by Mr. J. W. L. Glaisher in the B. A. Report for 1881. We easily find the following values:

$$\frac{1}{\mathrm{sn}^2 u} = k^2\, \mathrm{sn}^2 u - \frac{d}{du}\frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}\,,$$

$$\frac{\mathrm{cn}^2 u}{\mathrm{sn}^2 u} = -\, \mathrm{dn}^2 u - \frac{d}{du}\frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}\,,$$

$$\frac{\mathrm{dn}^2 u}{\mathrm{sn}^2 u} = k^2\, \mathrm{sn}^2 u - k^2 - \frac{d}{du}\frac{\mathrm{cn}\, u\, \mathrm{dn}\, u}{\mathrm{sn}\, u}\,.$$

$$\frac{\mathrm{sn}^2 u}{\mathrm{cn}^2 u} = \frac{1}{1+k^2}\left\{ \mathrm{dn}^2 u + \frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{dn}\, u}{\mathrm{cn}\, u} \right\},$$

$$\frac{1}{\mathrm{cn}^2 u} = \frac{k^2}{1+k^2} + \frac{k^2}{1+k^2}\, \mathrm{sn}^2 u + \frac{1+2k^2}{1+k^2}\frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{dn}\, u}{\mathrm{cn}\, u}\,,$$

$$\frac{\mathrm{dn}^2 u}{\mathrm{cn}^2 u} = k^2\, \mathrm{sn}^2 u + \frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{dn}\, u}{\mathrm{cn}\, u}\,,$$

$$\frac{\mathrm{sn}^2 u}{\mathrm{dn}^2 u} = \frac{1}{k'^2}\, \mathrm{cn}^2 u\, \mathrm{dn} - \frac{1}{k'^2}\frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u}$$

$$\frac{\mathrm{cn}^2 u}{\mathrm{dn}^2 u} = \mathrm{sn}^2 u + \frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u}$$

$$\frac{1}{\mathrm{dn}^2 u} = \frac{1}{k'^2} - \frac{k^2}{k'^2}\, \mathrm{sn}^2 u - \frac{k^2}{k'^2}\frac{d}{du}\frac{\mathrm{sn}\, u\, \mathrm{cn}\, u}{\mathrm{dn}\, u}\,.$$

The integrals of these quantities are dependent then upon the integral of $\mathrm{sn}^2 u$. Taking no account of signs or numerical factors we see that in the first group of three equations, which has as the denominator on the left-hand side the quantity $\operatorname{sn}^2 u$, the exact differential is $\frac{d^2}{du^2}\log\operatorname{sn} u$; in the second group where the corresponding denominator is $\operatorname{cn}^2 u$ the exact differential is $\frac{d^2}{du^2}\log\operatorname{cn} u$; and also in the third group with denominator $\operatorname{dn}^2 u$ the differential is $\frac{d^2}{du^2}\log\operatorname{dn} u$. The complete set of integrals is

$$\int\frac{du}{\operatorname{sn}^2 u}=\left\{\left(1-\frac{E}{K}\right)u-\frac{\Theta'(u)}{\Theta(u)}\right\}-\frac{d}{du}\log\operatorname{sn} u,$$

$$\int\frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u}du=-\left\{\frac{E}{K}u+\frac{\Theta'(u)}{\Theta(u)}\right\}-\frac{d}{du}\log\operatorname{sn} u,$$

$$\int\frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u}du=\left\{\left(k'^2-\frac{E}{K}\right)u-\frac{\Theta'(u)}{\Theta(u)}\right\}-\frac{d}{du}\log\operatorname{sn} u.$$

$$\int\frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u}du=\frac{1}{1+k^2}\left\{\frac{E}{K}u+\frac{\Theta'(u)}{\Theta(u)}\right\}-\frac{1}{1+k^2}\frac{d}{du}\log\operatorname{cn} u,$$

$$\int\frac{du}{\operatorname{cn}^2 u}=\frac{k^2}{1+k'^2}\left\{u+\frac{u}{k^2}\left(1-\frac{E}{K}\right)-\frac{1}{k^2}\frac{\Theta'(u)}{\Theta(u)}\right\}-\frac{1+2k^2}{1+k^2}\frac{d}{du}\log\operatorname{cn} u,$$

$$\int\frac{\operatorname{dn}^2 u}{\operatorname{cn}^2 u}du=\left\{\left(1-\frac{E}{K}\right)u-\frac{\Theta'(u)}{\Theta(u)}\right\}-\frac{d}{du}\log\operatorname{cn} u,$$

$$\int\frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}du=\frac{1}{k'^2}u-\frac{1}{k^2k'^2}\left\{\left(\frac{E}{K}-1\right)u+\frac{\Theta'(u)}{\Theta(u)}\right\}+\frac{1}{k^2k'^2}\frac{d}{du}\log\operatorname{dn} u,$$

$$\int\frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}du=\frac{1}{k^2}\left\{\left(1-\frac{E}{K}\right)u-\frac{\Theta'(u)}{\Theta(u)}\right\}-\frac{1}{k^2}\frac{d}{du}\log\operatorname{dn} u,$$

$$\int\frac{1}{\operatorname{dn}^2 u}du=\frac{1}{k'^2}\left\{\frac{E}{K}u+\frac{\Theta'(u)}{\Theta(u)}\right\}+\frac{1}{k^2}\frac{d}{du}\log\operatorname{dn} u.$$

Of course the whole set of integrals might be given in terms of the Θ-functions only, by substituting for $\frac{d}{du}\log\operatorname{sn} u$, $\frac{d}{du}\log\operatorname{cn} u$, $\frac{d}{du}\log\operatorname{dn} u$ their values, but it is not worth while to write them down. It is not difficult to obtain the integrals of the quantities

$$\operatorname{sn}^2 u\operatorname{cn}^2 u,\quad \operatorname{cn}^2 u\operatorname{dn}^2 u,\quad \operatorname{dn}^2 u\operatorname{sn}^2 u;$$

in fact it is obvious that they will all depend upon the integral

$$\int_0 \operatorname{sn}^4 u\, du,$$

and this can be shown to depend upon the integral of $\operatorname{sn}^2 u$.

The above reference to Mr. Glaisher's paper was made from memory, but I find on referring to it that he remarks that the integral of $\frac{1}{\operatorname{sn}^2 u}$ may be deduced from that of $\operatorname{sn}^2 u$ by the formula

$$\frac{d^2}{du^2} \log \operatorname{sn} u = k^2 \operatorname{sn}^2 u - \frac{1}{\operatorname{sn}^2 u};$$

he further shows in general how the integral of $\operatorname{sn}^n u$ may be obtained. I may perhaps be permitted to quote the last few lines of Mr. Glaisher's paper, which bear directly upon this question. In connection with the formulae of reduction for $\int \operatorname{sn}^n u \, du$, Mr. Glaisher says: "We have

1. $\frac{d^2}{du^2} \operatorname{sn}^n u = u(u-1) \operatorname{sn}^{n-2} u - u^2 (1 + k^2) \operatorname{sn}^n u + u(u+1) k^2 \operatorname{sn}^{n+2} u \ldots$

and by means of this formula the integral of $\operatorname{sn}^n u$ may be reduced to depend upon the integrals of $\operatorname{sn} u \operatorname{sn}^2 u$, $\frac{1}{\operatorname{sn} u}$, $\frac{1}{\operatorname{sn}^2 u}$ according as u is positive and uneven, positive and even, negative and uneven, negative and even.

The integral of $\operatorname{sn}^2 u$ involves the function $Z(u)$" $\left(i.\ e.\ \text{as above} \frac{\Theta'(u)}{\Theta(u)}\right)$; "and the integral of $\frac{1}{\operatorname{sn}^2 u}$ may be deduced from that of $\operatorname{sn}^2 u$ by the formula

2. $$\frac{d^2}{du^2} \log \operatorname{sn} u = k^2 \operatorname{sn}^2 u - \frac{1}{\operatorname{sn}^2 u}.$$

Corresponding to (1) and (2) there are eleven other pairs of formulae which involve the other eleven functions in place of $\operatorname{sn} u$, and differ from one another only in the k-coefficients. *It can thus be shown, that the integrals of the* n^{th} *powers of the twelve functions are all finitely expressible in terms of elliptic functions if* u *is uneven, and in terms of elliptic functions, and of the Zeta function if* u *is even; and that the twelve formulae of reduction are similar in form and deducible from any one of them.*"

The italics are my own. Mr. Glaisher has given in his paper the values of the integrals of the group of twelve primary functions marked (C) in the above.

It may be as well to work out the integral of $\operatorname{sn}^4 u$, as by means of it we can find at once the integrals of

$$\operatorname{cn}^4 u, \ \operatorname{dn}^4 u, \ \operatorname{sn}^2 u, \ \operatorname{cn}^2 u, \ \operatorname{cn}^2 u \operatorname{dn}^2 u, \ \operatorname{dn}^2 u \operatorname{sn}^2 u, \ \&c.$$

Applying Mr. Glaisher's general formula for the second differential coefficient of $\operatorname{sn}^n u$ with respect to u we have

$$\frac{d^2}{du^2}\operatorname{sn}^2 u = 2 - 4(1+k^2)\operatorname{sn}^2 u + 6k^2\operatorname{sn}^4 u.$$

From this on integration and reduction follows

$$\int_0 \operatorname{sn}^4 u\,du = \frac{1}{3k^2}\left\{\left[1+(1+k^2)\left(1-\frac{2E}{K}\right)\right]u - \frac{2(1+k^2)}{k^2}\frac{\Theta' u}{\Theta u} + \operatorname{sn} u\operatorname{cn} u\operatorname{dn} u\right\}.$$

This taken with the formula for $\int \operatorname{sn}^2 u\,du$ serves at once for the reduction of $\int \operatorname{cn}^4 u\,du$, $\int \operatorname{dn}^4 u\,du$, $\int \operatorname{sn}^2 u\operatorname{cn}^2 u\,du$, &c.

For the integral of $\frac{1}{\operatorname{sn}^2 u\operatorname{cn}^2 u}$ it is not necessary to use the above formula, for multiplying this by $\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1$ it reduces to the sum of the two integrals

$$\int_0 \frac{du}{\operatorname{cn}^2 u} + \int_0 \frac{du}{\operatorname{sn}^2 u}$$

the values of which are given above; on substituting these values we have

$$\int_0 \frac{du}{\operatorname{sn}^2 u\operatorname{cn}^2 u} = \left[2 - \frac{E}{K}\left(\frac{2+k^2}{1+k^2}\right)\right]u - \frac{\Theta'(u)}{\Theta(u)}\left(\frac{2+k^2}{1+k^2}\right)$$

$$-\frac{k^2}{1+k^2}\frac{d}{du}\log\operatorname{cn} u - \frac{d}{du}\log\operatorname{sn} u\operatorname{cn} u$$

or

$$= \left[2 - \frac{E}{K}\left(\frac{2+k^2}{1+k^2}\right)\right]u - \frac{\Theta'(u)}{\Theta(u)}\left(\frac{2+k^2}{1+k^2}\right)$$

$$-\frac{d}{du}\log\operatorname{sn} u\,(\operatorname{cn} u)^{\frac{1+2k^2}{1+k^2}}.$$

In like manner since

$$\frac{1}{\operatorname{sn}^2 u\operatorname{dn}^2 u} = \frac{1}{\operatorname{sn}^2 u} + \frac{k^2}{\operatorname{dn}^2 u}$$

we have

$$\int_0 \frac{du}{\operatorname{sn}^2 u\operatorname{dn}^2 u} = \left[1 - \frac{1-2k^2}{k'^2}\frac{E}{K}\right]u - \frac{1-2k^2}{k'^2}\frac{\Theta'(u)}{\Theta(u)}$$

$$+\frac{k^2}{k'^2}\frac{d}{du}\log\operatorname{dn} u - \frac{d}{du}\log\operatorname{sn} u$$

or

$$= \left[1 - \frac{1-2k^2}{k'^2}\,\frac{E}{K}\right]u - \frac{1-2k^2}{k'^2}\,\frac{\Theta'(u)}{\Theta(u)}$$

$$-\frac{d}{du}\log \operatorname{sn} u\,(\operatorname{dn} u)^{\frac{k^2}{k'^2}}$$

and also since

$$\frac{1}{\operatorname{cn}^2 u\,\operatorname{dn}^2 u} = \frac{1}{k'^2}\left(\frac{1}{\operatorname{cn}^2 u} - \frac{k^2}{\operatorname{dn}^2 u}\right)$$

we have

$$\int_0 \frac{du}{\operatorname{cn}^2 u\,\operatorname{dn}^2 u} = \frac{u}{k'^4(1+k^2)}\left[1 - k^4 - (1+k^4)\,\frac{E}{K}\right] - \frac{1}{k'^4}\,\frac{1+k^4}{1+k^2}\,\frac{\Theta'(u)}{\Theta(u)}$$

$$-\frac{1}{k'^2}\,\frac{1+2k^2}{1+k^2}\,\frac{d}{du}\log \operatorname{cn} u + \frac{1}{k'^2}\,\frac{d}{du}\log \operatorname{dn} u$$

or

$$= \frac{u}{k^4(1+k^2)}\left[1 - k^4 - (1+k^4)\,\frac{E}{K}\right] - \frac{1}{k'^4}\,\frac{1+k^4}{1+k^2}\,\frac{\Theta'(u)}{\Theta(u)}$$

$$-\frac{1}{k'^2}\,\frac{d}{du}\log\frac{(\operatorname{cn} u)^{\frac{1+2k^2}{1+k^2}}}{\operatorname{dn} u}.$$

A further set not mentioned in Mr. Glaisher's paper, nor in Prof. Cayley's Treatise, are the following three integrals

$$\int_0 \frac{\operatorname{sn} u}{\operatorname{cn} u\,\operatorname{dn} u}\,du\,,\quad \int_0 \frac{\operatorname{cn} u}{\operatorname{sn} u\,\operatorname{dn} u}\,du\,,\quad \int_0 \frac{\operatorname{dn} u}{\operatorname{sn} u\,\operatorname{cn} u}\,du$$

or

$$\int_0 \frac{du}{\frac{d}{du}\log \operatorname{sn} u}\,,\qquad -\int_0 \frac{du}{\frac{d}{du}\log \operatorname{cn} u}\,,\qquad -k^2\int_0 \frac{du}{\frac{d}{du}\log \operatorname{dn} u}\,.$$

These are very easily found; take for example the first one $\int \frac{\operatorname{sn} u}{\operatorname{cn} u\,\operatorname{dn} u}\,du$: write $\operatorname{sn} u = x$, then the integral is

$$\int \frac{x\,dx}{1-x^2.1-k^2x^2}\,.$$

Make

$$\frac{x}{1-x^2.1-k^2x^2} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{1-kx} + \frac{D}{1+kx}$$

and we find at once

$$A,\ B,\ C,\ D = \frac{1}{2k'^2}\,,\ -\frac{1}{2k'^2}\,,\ \frac{-k}{2k'^2}\,,\ \frac{k}{2k'^2}\,.$$

Substituting these values we have

$$\int \frac{x dx}{1-x^2 . 1-k^2x^2} = \frac{1}{2k'^2} \log \frac{1-k^2x^2}{1-x^2}.$$

Replacing x by its value gives

$$\int_0 \frac{\operatorname{sn} u}{\operatorname{cn} u \operatorname{dn} u} du = \frac{1}{k'^2} \log \frac{\operatorname{dn} u}{\operatorname{cn} u};$$

and similarly

$$\int_0 \frac{\operatorname{cn} u}{\operatorname{sn} u \operatorname{dn} u} du = \log \frac{\operatorname{sn} u}{\operatorname{dn} u},$$

$$\int_0 \frac{\operatorname{dn} u}{\operatorname{sn} u \operatorname{cn} u} du = - \log \frac{\operatorname{cn} u}{\operatorname{sn} u}.$$

From any two of these we obtain the third by addition. The similar integrals involving the squares of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ will involve the Θ-functions. We have, viz:—

$$\int_0 \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u \operatorname{dn}^2 u} du = \frac{1}{k'^2} \int \left(\frac{1}{\operatorname{cn}^2 u} - \frac{1}{\operatorname{dn}^2 u} \right) du$$

$$\int_0 \frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u \operatorname{dn}^2 u} du = \int_0 \left(\frac{1}{\operatorname{sn}^2 u} - \frac{k'^2}{\operatorname{dn}^2 u} \right) du$$

$$\int_0 \frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u \operatorname{cn}^2 u} du = \int_0 \left(\frac{1}{\operatorname{sn}^2 u} + \frac{k'^2}{\operatorname{cn}^2 u} \right) du.$$

Substituting from the above values of these quantities, we find readily

$$\int_0 \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u \operatorname{dn}^2 u} du = - \frac{1}{(1+k^2)(1-k^2)^2} \left[1 + k^4 + 2 \left(1 - \frac{E}{K} \right) u \right] + \\ \frac{2}{(1+k^2)(1-k^2)^2} \frac{\Theta'(u)}{\Theta(u)} - \frac{1}{1-k^2} \frac{d}{du} \log \left[\operatorname{cn} u^{\frac{1+2k^2}{1+k^2}} \operatorname{dn} u^{\frac{1}{k'^2}} \right]$$

$$\int_0 \frac{\operatorname{cn}^2 u}{\operatorname{sn}^2 u \operatorname{dn}^2 u} du = - \left[\frac{E}{K} u + \frac{\Theta'(u)}{\Theta(u)} \right] + \frac{d}{du} \log \frac{\operatorname{sn} u}{\operatorname{dn} u}$$

$$\int_0 \frac{\operatorname{dn}^2 u}{\operatorname{sn}^2 u \operatorname{cn}^2 u} du = \frac{1}{1+k^2} \left\{ (2 + 5k^2 + k^4) - 2(1+k^2)^2 \frac{E}{K} \right\} u \\ - 2(1+k^2) \frac{\Theta'(u)}{\Theta(u)} - \frac{d}{du} \log \frac{\operatorname{sn} u}{(\operatorname{cn} u)^{\frac{1+k^2-2k^4}{1+k^2}}}.$$

Of course, one could go on and extend these integrals indefinitely, but all subsequent forms would only depend upon those given in the above and in

Mr. Glaisher's paper. It may be worth while, however, to give the formulae for the integration of the quantities

$$\mathrm{sn}^{\alpha} u \; \mathrm{cn}^{\beta} u$$
$$\mathrm{cn}^{\alpha} u \; \mathrm{dn}^{\beta} u$$
$$\mathrm{dn}^{\alpha} u \; \mathrm{sn}^{\beta} u$$

in the several cases where α and β are either both even, one even and the other odd, or both odd, *i. e.* four cases in all for each of the three products. Suppose α and β both even, say $\alpha = 2n$, $\beta = 2m$:

Then

$$\mathrm{sn}^{2n} u \; \mathrm{cn}^{2n} u = \mathrm{sn}^{2n} u \, (1 - \mathrm{sn}^2 u)^m .$$

Again make $\alpha = 2n + 1$, $\beta = 2m$: then

$$\mathrm{sn}^{2n+1} u \; \mathrm{cn}^{2m} u = \mathrm{sn}^{2n+1} u \, (1 - \mathrm{sn}^2 u)^m.$$

Again make $\alpha = 2n$, $\beta = 2m + 1$: then

$$\mathrm{sn}^{2n} u \; \mathrm{cn}^{2m+1} u = \mathrm{cn}^{2m+1} u \, (1 - \mathrm{cn}^2 u)^n.$$

For these three cases we have

1. $$\int_0 \mathrm{sn}^{2n} u \; \mathrm{cn}^{2m} u \, du = \sum_{h=0}^{h=m} (-)^h \frac{m(m-1) \ldots (m-h+1)}{h!} \int_0 \mathrm{sn}^{2(h+n)} u \, du$$

2. $$\int_0 \mathrm{sn}^{2n+1} u \; \mathrm{cn}^{2m} u \, du = \sum_{h=0}^{h=m} (-)^h \frac{m(m-1) \ldots (m-h+1)}{h!} \int_0 \mathrm{sn}^{2(h+n+\frac{1}{2})} u \, du$$

3. $$\int_0 \mathrm{sn}^{2n} u \; \mathrm{cn}^{2m+1} u \, du = \sum_{h=0}^{h=m} (-)^h \frac{n(n-1) \ldots (n-h+1)}{h!} \int_0 \mathrm{cn}^{2(h+m+\frac{1}{2})} u \, du$$

and for the case of both odd we have

4. $$\int_0 \mathrm{sn}^{2n+1} u \; \mathrm{cn}^{2m+1} u \, du = \sum_{h=0}^{h=m} (-)^h \frac{m(m-1) \ldots (m-h+1)}{h!} \int_0 \mathrm{sn}^{2(h+n+\frac{1}{2})} u \; \mathrm{cn}\, u \, du .$$

Each term in (4) is of the form

$$\int_0 \mathrm{sn}^{2j+1} u \; \mathrm{cn}\, u \, du ;$$

this is

$$-\frac{1}{k^2}\int_0 \mathrm{sn}^{2j} u\, d(\mathrm{dn}\, u) = -\frac{1}{k^{2(j+1)}}\int_0 (1-\mathrm{dn}^2 u)^j\, d\, \mathrm{dn}\, u$$

$$= -\frac{1}{k^{2(j+1)}}\sum_{l=0}^{l=j}(-)^l\frac{j(j-1)\ldots(j-l+1)}{l!}\int \mathrm{dn}^{2l} u\, d(\mathrm{dn}\, u)$$

$$= -\frac{1}{k^{2(j+1)}}\sum_{l=0}^{l=j}(-)^l\frac{j(j-1)\ldots(j-l+1)}{l!\,(2l+1)}\,\mathrm{dn}^{2l+1} u\,.$$

Giving j the value that it has in (4), viz.

$$j = 2h + 2n + 1$$

we have for the integral in the general term of (4)

$$\int_0 \mathrm{sn}^{2(h+n+\frac{1}{2})} u\, du = -\frac{1}{k^{2(j+1)}}\sum_{l=0}^{l=2(h+n+\frac{1}{2})}(-)^l\frac{(2h+2n+1)(2h+2n)\ldots(2h+2n-l+2)}{l!\,(2l+1)}\, du^{2l+1} u$$

and so

4. $$\int_0 \mathrm{sn}^{2\ \ +1} u\, \mathrm{cn}^{2m+1} u\, du$$

$$= -\frac{1}{k^{2(j+1)}}\sum_{h=0}^{h=m}(-)^h\frac{m(m-1)\ldots(m-h+1)}{h!}$$
$$\sum_{l=\theta}^{l=2h+2n+1}(-)^l\frac{(2h+2n+1)(2h+2n)\ldots(2h+2n-l+2)}{l!\,(2l+1)}\,\mathrm{dn}^{2l+1} u\,.$$

By aid of the relations

$$k'^2(1+\tau^2\mathrm{cn}^2 u) = \mathrm{dn}^2 u$$

$$-\frac{1}{\tau^2}\left(1-\frac{\mathrm{dn}^2 u}{k'^2}\right) = \mathrm{cn}^2 u \qquad \tau^2 = \frac{k^2}{k'^2}$$

we readily find the integrals of

$$\mathrm{cn}^\alpha u\ \ \mathrm{dn}^\beta u$$

for the various odd and even values of α and β. These are

1′. $$\int_0 \mathrm{cn}^{2n} u\, \mathrm{dn}^{2m} u\, du = k'^{2m}\sum_{h=0}^{h=m}\frac{m(n-1)\ldots(m-h+1)}{h!}\,\tau^{2h}\int_0 \mathrm{cn}^{2(h+n)} u\, du,$$

2′. $$\int_0 \mathrm{cn}^{2n+1} u\, \mathrm{dn}^{2m} u\, du = k'^{2m}\sum_{h}^{h=m}\frac{m(m-1)\ldots(m-h+1)}{h!}\,\tau^{2h}\int \mathrm{cn}^{2(h+n)} u\, du,$$

$$3'.\quad \int_0 \mathrm{cn}^{2n} u\, \mathrm{dn}^{2m+1} u\, du = (-)^n \frac{1}{\tau^{2n}} \sum_{h=0}^{h=n} (-)^n \frac{n(n-1)\ldots(n-h+1)}{h!\, k'^{2h}} \int_0 \mathrm{dn}^{2h} u\, du,$$

$$4'.\quad \int_0 \mathrm{cn}^{2n+1} u\, \mathrm{dn}^{2m+1} u\, du =$$

$$-k'^2 \sum_{h=0}^{h=n} \frac{m(m-1)\ldots(m-h+1)}{h!} \tau^{2h} \sum_{l=0}^{l=2h+2n+1} (-)^l \frac{(2h+2n+1)\ldots(2h+2n-l+2)}{l!\,(2l+1)} \mathrm{sn}^{2l+1} u$$

Again since

$$\mathrm{sn}^2 u = \frac{1}{k^2}(1-\mathrm{dn}^2 u),$$

we have

$$1''.\quad \int_0 \mathrm{dn}^{2n} u\, \mathrm{sn}^{2m} u\, du = \frac{1}{k^{2m}} \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!} \int_0 \mathrm{dn}^{2(h+n)} u\, du,$$

$$2''.\int_0 \mathrm{dn}^{2n+1} u\, \mathrm{sn}^{2m} u\, du = \frac{1}{k^{2m}} \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!} \int_0 \mathrm{dn}^{2(h+n\quad\frac{1}{2})} u\, du,$$

$$3''.\int_0 \mathrm{dn}^{2n} u\, \mathrm{sn}^{2m+1} u\, du = \sum_{h=0}^{h=n} (-)^h \frac{n(n-1)\ldots(n-h+1)}{h!} k^{2h} \int_0 \mathrm{sn}^{2(h+m+\frac{1}{2})} u\, du,$$

$$4''.\quad \int_0 \mathrm{dn}^{2n+1} u\, \mathrm{sn}^{2m+1} u\, du = \frac{1}{k^{2m}} \sum_{h=0}^{h=m} (-)^h \frac{m(m-1)\ldots(m-h+1)}{h!} (-k'^{2h+2n+1})$$

$$\sum_{l=0}^{l=2h+2n+1} \frac{(2h+2n+1)\ldots(2h+2n-l+2)}{l!\,(2l+1)} \tau^{2l}\, \mathrm{cn}^{2l+1} u.$$

The formulae 2, 3, 2′, 3′, 2″, 3″ might have been given in forms similar to 4, 4′, 4″. For the integrations in these sets of equations we need only to employ the formulae

$$\frac{d^2}{du^2} \mathrm{sn}^n u = n(n-1)\,\mathrm{sn}^{n-2} u - n^2(1+k^2)\,\mathrm{sn}^n u + n(n+1)\,k^2\,\mathrm{sn}^{n+2} u,$$

$$\frac{d^2}{du^2} \mathrm{cn}^n u = n(n-1)\,k'^2\,\mathrm{cn}^{n-2} u + n^2(k^2-k'^2)\,\mathrm{cn}^n u - n(n+1)\,k^2\,\mathrm{cn}^{n+2} u,$$

$$\frac{d^2}{du^2} \mathrm{dn}^n u = -n(n-1)\,k'^2\,\mathrm{dn}^{n-2} u + n^2(1+k'^2)\,\mathrm{dn}^n u - n(n+1)\,\mathrm{dn}^{n+2} u.$$

I have tried to find reasonably simple and symmetrical forms for the expansions of the right-hand members of the above equations, by grouping together all terms containing only elliptic functions, and all containing the Θ-function, but, though I think such forms should exist, I have not been able to hit upon them. Another set of formulae might be found for the integrals of

$$\frac{1}{\mathrm{sn}^{\alpha} u\, \mathrm{cn}^{\beta} u}, \quad \frac{1}{\mathrm{cn}^{\alpha} u\, \mathrm{dn}^{\beta} u}, \quad \frac{1}{\mathrm{dn}^{\alpha} u\, \mathrm{sn}^{\beta} u},$$

$$\frac{\mathrm{sn}^{\alpha} u\, \mathrm{cn}^{\beta} u}{\mathrm{dn}^{\gamma} u}, \quad \frac{\mathrm{cn}^{\alpha} u\, \mathrm{dn}^{\beta} u}{\mathrm{sn}^{\gamma} u}, \quad \frac{\mathrm{dn}^{\alpha} u\, \mathrm{sn}^{\beta} u}{\mathrm{cn}^{\gamma} u},$$

$$\frac{\mathrm{dn}^{\gamma} u}{\mathrm{sn}^{\alpha} u\, \mathrm{cn}^{\beta} u}, \quad \frac{\mathrm{sn}^{\gamma} u}{\mathrm{cn}^{\alpha} u\, \mathrm{dn}^{\beta} u}, \quad \frac{\mathrm{cn}^{\gamma} u}{\mathrm{dn}^{\alpha} u\, \mathrm{sn}^{\beta} u},$$

but they are easily deducible from the preceding and following forms.

The formulae of integration for the several cases involved in the expression

$$\int_0 \mathrm{sn}^{\alpha'} u\, \mathrm{cn}^{\beta'} u\, \mathrm{dn}^{\gamma'} u\, du$$

are very easily found; there are in all 8 cases, viz.

	α',	β',	γ',		α',	β',	γ',
1.	2α,	2β,	2γ,	5.	$2\alpha+1$,	$2\beta+1$,	2γ,
2.	$2\alpha+1$,	2β,	2γ,	6.	2α,	$2\beta+1$,	$2\gamma+1$,
3.	2α,	$2\beta+1$,	2γ,	7.	$2\alpha+1$,	2β,	$2\gamma+1$,
4.	2α,	2β,	$2\gamma+1$,	8.	$2\alpha+1$,	$2\beta+1$,	$2\gamma+1$.

For brevity write

$$[\zeta, \lambda] = \frac{\zeta(\zeta-1)\ldots(\zeta-\lambda+1)}{\lambda!}.$$

It is very easy now to find the integral of $\mathrm{sn}^{\alpha'} u\, \mathrm{cn}^{\beta'} u\, \mathrm{dn}^{\gamma'} u$ for the above 8 cases by aid of the relations

$$1-\mathrm{sn}^2 u = \mathrm{cn}^2 u, \quad 1-k^2\mathrm{sn}^2 u = \mathrm{dn}^2 u, \quad 1-\mathrm{cn}^2 u = \mathrm{sn}^2 u$$

$$-\frac{1}{\tau^2}\left(1-\frac{\mathrm{dn}^2 u}{k'^2}\right) = \mathrm{cn}^2 u, \quad k'^2(1+\tau^2\mathrm{cn}^2 u) = \mathrm{dn}^2 u, \quad \frac{1}{k^2}(1-\mathrm{dn}^2 u) = \mathrm{sn}^2 u$$

where as above $\tau^2 = \frac{k^2}{k'^2}\cdot$ The integrals are

$$1. \quad \int_0 \mathrm{sn}^{2\alpha} u\, \mathrm{cn}^{2\beta} u\, \mathrm{dn}^{2\gamma} u\, du = \sum_{h=0}^{h=\beta}\sum_{l=0}^{l=\beta}(-)^{h+l}[\beta, h][\gamma, l]k^{2l}\int \mathrm{sn}^{2(h+l+\alpha)} u\, du$$

2. $$\int_0 \mathrm{sn}^{2\alpha+1}u\ \mathrm{cn}^{2\beta}u\ \mathrm{dn}^{2\gamma}u = \sum_{h=0}^{h=\beta}\sum_{l=0}^{l=\gamma}[\beta,\ h][\gamma,\ l]\,k^{2l}\int \mathrm{sn}^{2(h+l+\alpha+\frac{1}{2})}u\,du$$

3. $$\int_0 \mathrm{sn}^{2\alpha}u\ \mathrm{cn}^{2\beta+1}u\ \mathrm{dn}^{2\gamma}u\,du = k'^{2\gamma}\sum_{h=0}^{h=\alpha}\sum_{l=0}^{l=\gamma}(-)^{h+l}[\alpha,\ h][\gamma,\ l]\tau^{2l}$$
$$\int_0 \mathrm{cn}^{2(h+l+\beta+\frac{1}{2})}u\,du$$

4. $$\int_0 \mathrm{sn}^{2\alpha}u\,\mathrm{cn}^{2\beta}u\ \mathrm{dn}^{2\gamma+1}u\,du = (-)^{\beta}\frac{1}{\tau^{2\beta}k^{2\alpha}}\sum_{h=0}^{h=\alpha}\sum_{l=0}^{l=\beta}\frac{[\alpha,\ h][\beta,\ l]}{k'^{2l}}\int_0 \mathrm{dn}^{2(h+l+\gamma+\frac{1}{2})}u\,du$$

5. $$\int_0 \mathrm{sn}^{2\alpha+1}u\ \mathrm{cn}^{2\beta+1}u\ \mathrm{dn}^{2\gamma}u\,du$$
$$= (-)^{\beta+1}\frac{1}{\tau^{2\beta}k^{2(\alpha+1)}}\sum_{h=0}^{h=\alpha}\sum_{l=0}^{l=\beta}(-)^{h+l}\frac{[\alpha,\ h][\beta,\ l]}{k'^{2l}}\cdot\frac{\mathrm{dn}^{2(h+l+\gamma+\frac{1}{2})}u}{2(h+l+\gamma+\frac{1}{2})}$$

6. $$\int_0 \mathrm{sn}^{2\alpha}u\ \mathrm{cn}^{2\beta+1}u\ \mathrm{dn}^{2\gamma+1}u\,du = \sum_{h=0}^{h=\beta}\sum_{l=0}^{l=\gamma}[\beta,\ h][\gamma,\ l]\,k^{2l}\frac{\mathrm{sn}^{2(h+l+\alpha+\frac{1}{2})}u}{2(h+l+\alpha+\frac{1}{2})}$$

7. $$\int_0 \mathrm{sn}^{2\alpha+1}u\ \mathrm{cn}^{2\beta}u\ \mathrm{dn}^{2\gamma+1}u\,du = -k'^{2\gamma}\sum_{h=0}^{h=\alpha}\sum_{l=0}^{l=\gamma}(-)^{h}[\alpha,\ h][\gamma,\ l]\tau^{2l}\frac{\mathrm{cn}^{2(h+l+\beta+\frac{1}{2})}u}{2(h+l+\beta+\frac{1}{2})}$$

8. $$\int_0 \mathrm{sn}^{2\alpha+1}u\ \mathrm{cn}^{2\beta+1}u\ \mathrm{dn}^{2\gamma+1}u\,du = \sum_{h=0}^{h=\beta}\sum_{l=0}^{l=\gamma}(-)^{h+l}[\beta,\ h][\gamma,\ l]k^{2l}\frac{\mathrm{sn}^{2(h+l+\alpha+1)}u}{2(h+l+\alpha+1)}.$$